Towards a classification of static electro–vacuum space–times containing an asymptotically flat spacelike hypersurface with compact interior

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Abstract

We show that static electro–vacuum black hole space–times containing an asymptotically flat spacelike hypersurface with compact interior and with both degenerate and non–degenerate components of the event horizon do not exist, under the supplementary hypothesis that all degenerate components of the event horizon have charges of the same sign. This extends previous uniqueness theorems of Simon [26] and Masood–ul–Alam [22] (where only non–degenerate horizons were allowed) and Heusler [19] (where only degenerate horizons were allowed).

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1 Introduction

A classical question in general relativity, first raised and partially answered by Israel [20], is that of classification of suitably regular static[1] black hole solutions of the Einstein–Maxwell equations. The most complete results existing in the literature so far are due to Simon [26], Masood–ul–Alam [22] and Heusler [17, 19] who show, roughly speaking, the following:

1. Suppose that all the horizons are non–degenerate. Then the black hole is a Reissner–Norsdström black hole [17, 22, 26].

2. Suppose instead that all the horizons are degenerate, and that

$$\forall\ i,j \qquad Q_iQ_j \geq 0\ , \tag{1.1}$$

where Q_i is the charge of the i–th connected component of the black hole. Then the black hole is a standard Majumdar–Papapetrou black hole [19] (*cf.* also [**?**, 9]).

Heusler's condition (1.1) is obviously satisfied by a connected black hole, so that the above results settle the classification question in the connected case (recall that a standard connected Majumdar–Papapetrou black hole is an extreme Reissner–Nordström one). The general case, however, remains still open. In this paper we "merge" those two results and show the following:

Theorem 1.1 *Let (M, g, F) be a solution of the Einstein–Maxwell equations containing a connected space-like hypersurface Σ, the closure $\bar{\Sigma}$ of which is the union of a finite number of asymptotically flat ends and of a compact interior. Let X be a Killing vector field on M which is timelike, future directed in all the asymptotically flat ends, which leaves F invariant and which satisfies the hypersurface–orthogonality condition (2.1). Suppose moreover that:*

1. *We have $g_{\mu\nu}X^\mu X^\nu < 0$ on[2] Σ.*

2. *The topological boundary $\partial\Sigma \equiv \overline{\Sigma} \setminus \Sigma$ of Σ is a nonempty topological manifold, with $g_{\mu\nu}X^\mu X^\nu = 0$ on $\partial\Sigma$.*

Then:

[1] A space–time (M, g) with Killing vector field X will be called static if X is (locally) hypersurface orthogonal everywhere, and if X is timelike for all sufficiently distant points in the relevant asymptotically flat regions, *cf.* Section 2. The regions of M where X is timelike are thus static in the usual sense. It should be emphasized that we allow the defining Killing vector of a static space–time to be spacelike in some regions. Thus both the Schwarzschild space–time and its Kruszkal–Szekeres extension are static in our terminology. We hope that this will not lead to confusions.

[2] We use the signature $(-, +, +, +)$.

1. *If* $\partial\Sigma$ *is* connected*, then* Σ *is diffeomorphic to* $\mathbb{R}^3$ *minus a ball. Moreover there exists a neighborhood of* Σ *in* M *which is isometrically diffeomorphic to an open subset of the (extreme or non–extreme) Reissner–Nordström space–time.*

2. *If* $\partial\Sigma$ *is* not *connected and if condition (1.1) holds for charges* Q_i *associated to those components of* $\partial\Sigma$ *that intersect* the degenerate horizons*, then* Σ *is diffeomorphic to* $\mathbb{R}^3$ *minus a finite union of disjoint balls. Moreover the space–time contains* only *degenerate horizons, and there exists a neighborhood of* Σ *in* M *which is isometrically diffeomorphic to an open subset of the standard Majumdar–Papapetrou space–time.*

Actually a somewhat more general result is proved in Theorem 3.6 below. We emphasize that no sign conditions are made concerning the charges of non–degenerate horizons. We also note that simple connectedness of Σ will hold when appropriate further global hypotheses on M are done, *cf.* Theorem 1.3 below. Thus, to obtain a satisfactory classification of the space–times under consideration it remains to remove the condition on the sign of the charges, or to construct (and classify) appropriately regular black holes which do not satisfy this condition. We find that last possibility rather unlikely.

The definitions and conventions used here coincide with those of the accompanying paper [8]. Those definitions which cannot be found there are presented in Section 2 below.

We refer the reader to a discussion of a similar theorem for vacuum space–times in [8, Section 1] for comments concerning the improvements of this result as compared to the ones available in the literature even in cases where a mixture of degenerate and non–degenerate horizons is forbidden. It might be of some interest to mention that our conclusion will still hold for quite a larger class of manifolds Σ. A possible generalization is that with Σ being *e.g.* the union of a) a finite number of asymptotically flat ends with b) a neighborhood of the boundary $\partial\Sigma$ which has compact closure in M and c) a non–compact region on which we have $0 < \epsilon \leq 1 + \phi \pm \sqrt{-g_{\mu\nu}X^\mu X^\nu}$, provided that Σ with the induced metric is a *complete* Riemannian manifold; ϕ here is the electric potential as defined in Equation (3.3) after the relevant duality rotations have been performed, *cf.* Lemma 3.2. The proof carries through without any modifications to this case.

Our strategy is a modification of that of Ruback[3] [25] along the lines of [8]: we consider the *orbit space* metric h on Σ, as defined in [8]. The key tool here are the results of [8] concerning the geometry of (Σ, h) near both the degenerate

[3]We note that while the relevant claims in [25] can be eventually justified, the paper [25] contains several essential gaps. The work here can be considered as an extension of that of Ruback to include degenerate black holes, together with a justification of the relevant unsubstantiated claims made in [25]. We further note that we have not been able to adapt the technique of Simon [26] and Masood–ul–Alam [22] to include degenerate black holes without having to introduce some supplementary restrictions.

components of $\partial\Sigma$ and the non–degenerate ones. Next, following [25], we consider a manifold which consists of two copies of (Σ, h) glued along all *non–degenerate* components of $\partial\Sigma$, equipped with an appropriate conformally deformed metric. As in [8] we use a new version of the positive energy theorem proved in [**?**] (Theorem 3.3 below) to show that the metric on Σ is conformally flat. One can then use classical calculations to finish the proof. We note that it is usual in the last step of the proof to invoke analyticity to conclude. Because analytic extensions of manifolds are not unique this is not sufficient without a more thorough justification. We finish the proof by a simple open–closed argument which avoids this problem.

Under the hypotheses of Theorem 1.1 there is no chance of getting more information about the size of the set on which the metric is that of a Reissner–Nordström or a standard Majumdar–Papapetrou space–time (consider any hypersurface Σ in the Reissner–Nordström space–time, and set M to be any neighborhood of Σ which does not coincide with the Reissner–Nordström space–time; alternatively, identify t with $t+1$ in the Reissner–Nordström space–time). In complete analogy with the vacuum case in [8] we have the following:

Corollary 1.2 *Under the hypotheses of Theorem 1.1, assume further that*

3. *The orbits of the Killing vector X through Σ are complete.*

Then the following properties are equivalent:

i. *$\Sigma_{\rm ext}$ is achronal*[4] *in $M_{\rm ext}$.*

ii. *$M_{\rm ext}$ is diffeomorphic to $\mathbb{R}\times\Sigma_{\rm ext}$ (which is equivalent to an appropriately complete $\mathscr{J}$ having $\mathbb{R}\times S^2$ topology).*

iii. *There are no closed timelike curves through $\Sigma_{\rm ext}$ contained in $M_{\rm ext}$.*

Further, if one (and hence all) of the above conditions holds, then the Killing development[5] *$\mathcal{K}(\Sigma)$ of Σ defined as*

$$\mathcal{K}(\Sigma) \equiv \cup_{t\in\mathbb{R}}\phi_t(\Sigma) \ , \tag{1.2}$$

where ϕ_t is the action of the isometry group generated by X, equipped with the induced metric, is isometrically diffeomorphic to a domain of outer communications of a standard extension of a Reissner–Nordström space–time or of a standard Majumdar–Papapetrou space–time.

[4] By that we mean that there are no timelike curves from $\Sigma_{\rm ext}$ to itself which are entirely contained in $M_{\rm ext}$.

[5] The notion of Killing development used here differs slightly from the definition given in [4], as we allow here a topology of $\mathcal{K}(\Sigma)$ which is not $\mathbb{R}\times\Sigma$.

The *standard Majumdar–Papapetrou* space–times are defined in Section 2. We refer the reader to the introduction of [8] for a discussion of the relationship between Theorem 1.1 and Corollary 1.2 and black holes. In particular in the introduction of [8] an example was given which shows that more hypotheses than those of Corollary 1.2 are needed to show that $\mathcal{K}(\Sigma)$ coincides with a d.o.c. in M. For reference we state the following:

Theorem 1.3 *Let (M, g, F) be a solution of the Einstein–Maxwell equations containing a connected space-like hypersurface Σ, the closure $\bar{\Sigma}$ of which is the union of a finite number of asymptotically flat ends and of a compact interior. Let X be a Killing vector field on M which is timelike future directed in all the asymptotically flat ends, and which satisfies the hypersurface–orthogonality condition (2.1). Let further $\mathcal{D}_{oc} \equiv \mathcal{D}_{oc}(M_{\text{ext}})$ be a domain of outer communications in (M, g) associated with one of the asymptotically flat ends of Σ. Suppose that:*

1. *We have $\Sigma \subset \mathcal{D}_{oc}$.*
2. *The topological boundary $\partial\Sigma \equiv \overline{\Sigma}\backslash\Sigma$ of Σ is a nonempty topological manifold and satisfies $\partial\Sigma = \overline{\Sigma} \cap \partial\mathcal{D}_{oc}$.*
3. *X has complete orbits in $\mathcal{D}_{oc}$.*

In addition to the above, suppose that condition (1.1) holds for charges Q_i associated to those components of $\partial\Sigma$ that intersect the degenerate horizons *and that one of the following conditions holds:*

4a) Either $(\mathcal{D}_{oc}, g|_{\mathcal{D}_{oc}})$ is globally hyperbolic, or

4b) (M, g) is globally hyperbolic.

Then the conclusions of Theorem 1.1 and Corollary 1.2 hold. Moreover $\mathcal{D}_{oc}$ is isometrically diffeomorphic to a domain of outer communications of a standard extension of a Reissner–Nordström space–time or of a standard Majumdar–Papapetrou space–time.

We note that it is not assumed above that X is timelike throughout Σ.

The proofs of both Corollary 1.2 and Theorem 1.3 are essentially identical to the corresponding ones in [8]; some comments about the proof of Corollary 1.2 can be found at the end of Section 3; the proof of Theorem 1.3 will be omitted. We note that the property that Σ is simply connected and has only one asymptotically flat end required in Theorem 1.1 follows from [10]. We further note that the obvious electro–vacuum generalization of the remaining cases of Theorem 1.3 of [8] holds under the supplementary hypothesis that Σ is simply connected and has only one asymptotically flat end.

This paper is organized as follows: Section 2 contains definitions and some preliminary remarks. In section 3 we prove Theorem 1.1, as a consequence of the somewhat more general Theorem 3.6, which is also proved there.

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2 Preliminaries

Our conventions and definitions are as in [8, Section 2]. Further, a triple (M, g, F) will be said to be static if there exists on M a Killing vector field X such that the Maxwell two–form field F satisfies

$$\mathcal{L}_X F = 0 \ ,$$

with X satisfying moreover the hypersurface–orthogonality condition:

$$X_{[\alpha}\nabla_\beta X_{\gamma]} = 0 \ . \tag{2.1}$$

Here and throughout $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

Next, a data set $(\Sigma_{\rm ext}, g, K)$ with Maxwell field F will be called an *asymptotically flat end* if $\Sigma_{\rm ext}$ is diffeomorphic to $\mathbb{R}^3$ minus a ball and if the fields (g_{ij}, K_{ij}) satisfy the fall–off conditions

$$|g_{ij}-\delta_{ij}|+r|\partial_\ell g_{ij}|+\cdots+r^k|\partial_{\ell_1\cdots\ell_k}g_{ij}|+r|K_{ij}|+\cdots+r^k|\partial_{\ell_1\cdots\ell_{k-1}}K_{ij}| \le C_{k,\alpha}r^{-\alpha} \ , \tag{2.2}$$

for some constants $C_{k,\alpha}$, $\alpha>0$, $k\ge 1$. We shall further require that in the local coordinates as above on $\Sigma_{\rm ext}$ the Maxwell field satisfies the fall–off conditions

$$|F_{\mu\nu}|+r|\partial_\ell F_{\mu\nu}|+\cdots+r^k|\partial_{\ell_1\cdots\ell_k}F_{\mu\nu}| \le \hat C_{k,\alpha}r^{-\alpha-1} \ , \tag{2.3}$$

for some constants $\hat C_{k,\alpha}$, $\alpha>0$, $k\ge 0$. We shall always implicitly assume $\alpha>1/2$ when the ADM mass will be invoked, as this condition makes it well defined in vacuum. It follows in any case from [7, Section 1.3] that in stationary electro–vacuum space–times there is no loss of generality in assuming $\alpha=1$, k – arbitrary. A hypersurface will be said to be *asymptotically flat* if it contains an asymptotically flat end $\Sigma_{\rm ext}$.

To avoid ambiguities, we define the *Reissner–Nordström* space–time $(M^{\rm RN}, g^{\rm RN})$ to be the manifold $\{t\in\mathbb{R}, r\in(m+\sqrt{m^2-Q^2-P^2},\infty), q\in S^2\}$, with

$$m^2-Q^2-P^2\ge 0 \ , \tag{2.4}$$

and with the metric

$$g^{\rm RN}=-(1-\frac{2m}{r}+\frac{Q^2+P^2}{r^2})dt^2+(1-\frac{2m}{r}+\frac{Q^2+P^2}{r^2})^{-1}dr^2+r^2d\Omega^2 \ , \tag{2.5}$$

where $d\Omega^2$ is the standard round metric on a unit two–dimensional sphere S^2. It is somewhat awkward to build in the inequality (2.4) in our definition of a

Reissner–Nordström space–time, but it saves us the need of repeating that (2.4) holds each time we mention a Reissner–Nordström space–time. The Maxwell field is

$$F \equiv F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{Q}{r^2} dt \wedge dr - P \sin(\theta) d\theta \wedge d\phi \ , \tag{2.6}$$

so that Q is the total electric charge and P is the total magnetic charge of $\Sigma_{\rm ext}$. We will refer to those coordinates as the standard coordinates on the Reissner–Nordström space–time. We shall call the *standard extension of the Reissner–Nordström space–time* the extension of $(M^{\rm RN}, g^{\rm RN})$ described *e.g.* by the Carter–Penrose diagram on page 158 of [15] for $m^2 > Q^2 - P^2$ and on page 160 of [15] for $m^2 = Q^2 - P^2$.

Recall that the Majumdar–Papapetrou (MP) metrics are, locally, of the form [21, 24]

$$g = -u^{-2} dt^2 + u^2 (dx^2 + dy^2 + dz^2) \, , \tag{2.7}$$

$$A = u^{-1} dt \, , \tag{2.8}$$

where A is the Maxwell potential, $F = dA$, with some nowhere vanishing, say positive, function u. A space–time will be called a standard MP space–time if the coordinates x^μ of (2.7)–(2.8) are global with range $\mathbb{R} \times (\mathbb{R}^3 \setminus \{\vec{a}_i\})$ for a finite set of points $\vec{a}_i \in \mathbb{R}^3$, $i = 1, \ldots, I$, and if the function u has the form

$$u = 1 + \sum_{i=1}^{I} \frac{m_i}{|\vec{x} - \vec{a}_i|} \ , \tag{2.9}$$

for some positive constants m_i. It has been shown by Hartle and Hawking [14] that every standard MP space–time can be analytically extended to an electro–vacuum space–time with a non–empty black hole region, and with a domain of outer communication which is non–singular in the sense of the theorems proved here. Those extensions will be called the *standard* extensions of the standard Majumdar–Papapetrou space–times.

3 Proof of Theorem 1.1

Following [8], we equip Σ with the orbit space metric h defined as

$$h(Y, Z) = g(Y, Z) - \frac{g(X, Y) g(X, Z)}{g(X, X)} \ , \tag{3.1}$$

where X is the defining Killing vector, that is, the Killing vector which asymptotes $\partial/\partial t$ in the asymptotic regions, and satisfies the hypersurface–orthogonality condition (2.1). Let the electric field E and the magnetic field B be defined on M by the equations (we use the conventions of [18])

$$E(Y) = -F(X, Y) \ , \qquad B(Y) = (*F)(X, Y) \ , \tag{3.2}$$

where $*F$ denotes the space–time Hodge dual of the Maxwell field two–form F. Simple connectedness of Σ and a standard calculation (*cf., e.g.*, [18]) show that there exist functions ϕ and ψ defined in a neighborhood of Σ in M such that we have

$$E = d\phi \ , \quad B = d\psi \ , \qquad \mathcal{L}_X\phi = \mathcal{L}_X\psi = 0 \ . \tag{3.3}$$

By an abuse of notation we shall often use the symbol ϕ to denote the restriction of ϕ to Σ, similarly with ψ. The potentials ϕ and ψ are of course defined up to a constant, and we can normalize them so that on $\Sigma_{\rm ext}$ we have

$$\phi = \frac{Q}{r} + O(r^{-2}) \ , \qquad \psi = \frac{P}{r} + O(r^{-2}) \ , \tag{3.4}$$

where Q is the total electric charge and P is the total magnetic charge in $\Sigma_{\rm ext}$. (We note if there were several asymptotically flat ends it could happen that the potentials could asymptote constants different from zero on some ends, and the proof given below would break down. This is the only place where the hypothesis that Σ has only one end enters in the argument. In fact, one could allow several ends when the supplementary hypothesis is made that ϕ and ψ can be normalized to asymptote to zero in all asymptotically flat ends.)

The metric h on Σ is essentially "the metric that would have been induced on Σ if Σ were normal to X", so that we have the following equivalent of Lemma 5.1 of [8], the proof of which is a repetition of that in [8]:

Lemma 3.1 *Suppose that (M, g, F) is static and assume that the set $(\hat{h}, \hat{V}, \hat{\phi}, \hat{\psi})$, where $\hat{h}$ is the metric induced on the hypersurfaces orthogonal to X, $-\hat{V}^2$ is the square of the Lorentzian norm of X on those hypersurfaces, and $\hat{\phi}$ (respectively $\hat{\psi}$) is the restriction of the electric potential ϕ (respectively the magnetic potential ψ) defined by Equations (3.2) and (3.3) to those hypersurfaces, satisfies some coordinate–independent system of equations. Then the orbit space–metric h together with the function V (such that $-V^2$ is the square of the Lorentzian norm of X on Σ), the electric potential $\phi|_\Sigma$ and the magnetic potential $\psi|_\Sigma$ satisfy the same system of equations.*

It follows that in the Einstein–Maxwell case we have the equations:

$$V\Delta_h\phi = h(d\phi, dV) \ , \tag{3.5}$$

$$V\Delta_h\psi = h(d\psi, dV) \ , \tag{3.6}$$

$$V\Delta_h V = h(d\phi, d\phi) + h(d\psi, d\psi) \ , \tag{3.7}$$

$$VR_{ij} = D_iD_jV + V^{-1}\{(h(d\phi, d\phi) + h(d\psi, d\psi))h_{ij} - 2\phi_{,i}\phi_{,j} - 2\psi_{,i}\psi_{,j}\} \ , \tag{3.8}$$

where Δ_h is the Laplace operator of the metric h, R_{ij} is the Ricci tensor of h, and where a comma denotes differentiation. In particular we have

$$R \equiv h^{ij}R_{ij} = 2V^{-2}\{h(d\phi, d\phi) + h(d\psi, d\psi)\} \ . \tag{3.9}$$

Following Heusler [17] we note:

Lemma 3.2 *Under the hypotheses of Theorem 1.1 the magnetic field B can be made to vanish by a duality rotation.*

PROOF: If $E \equiv 0$ the result is obvious by exchanging ϕ with ψ. Suppose thus that E is not identically vanishing, as shown *e.g.* in [17] we then have

$$B = \mu E \ ,$$

with μ being constant on each connected component of the set $\Omega \equiv \{E \neq 0\}$. Let Ω_0 be any connected component of Ω, by performing a duality rotation we can obtain $\psi = 0$ in Ω_0 [17]. As Ω_0 is open, Equation (3.6) and the unique continuation theorem of Aronszajn [1] show that $\psi \equiv 0$, hence $B \equiv 0$. ∎

Unless explicitly stated otherwise, in the remainder of the paper we shall assume that the duality transformation of Lemma 3.2 has been performed, so that

$$\psi \equiv 0 \ .$$

In the proof of Theorem 1.1 we shall need the following version of the positive energy theorem, proved in [**?**]:

Theorem 3.3 *Let $(\hat\Sigma, \hat h)$ be a smooth complete Riemannian manifold with an asymptotically flat end $\hat\Sigma_{\mathrm{ext}}$ (in the sense of Equation (2.2) with $k \geq 4$ and $\alpha > 1/2$) and with a smooth divergence free vector field $\hat E$ satisfying*

$$\hat E_i dx^i = \frac{\hat Q}{r^2} dr + o(r^{-2})$$

in $\hat\Sigma_{\mathrm{ext}}$. Suppose that the Ricci scalar $\hat R$ of $\hat h$ satisfies

$$0 \leq \hat R - 2\hat h(\hat E, \hat E) \in L^1(\hat\Sigma_{\mathrm{ext}}) \ .$$

Then the ADM mass $\hat m$ of $\hat\Sigma_{\mathrm{ext}}$ satisfies

$$\hat m \geq |\hat Q| \ ,$$

*where $\hat Q$ is the total charge of $\hat\Sigma_{\mathrm{ext}}$. If the equality is attained and $\hat E$ is not identically vanishing, then the metric $\hat h$ is, locally, the metric induced on the $t =$const slices of a Majumdar–Papapetrou space–time (*cf. *Equation (2.7)) with*

$$\hat E_i dx^i = \frac{du}{u} \ , \tag{3.10}$$

where u is as in (2.7)–(2.8).

We emphasize that in the result above $\hat{\Sigma}$ can have an arbitrary number (perhaps infinite) of asymptotic ends, and that *no hypotheses* are made on the asymptotic behavior of the metric in those ends except that the metric $\hat{h}$ is complete (and that at least one of the ends is asymptotically flat so that its ADM mass is well defined). More general results, allowing for non–vanishing extrinsic curvature of the initial data hypersurface, non–vanishing of the magnetic field, poor differentiability of the metric, and boundaries, can be found in [**?**]. The proof of Theorem 3.3 uses a Witten–type spinorial argument based on the suggestion of Gibbons and Hull [11]. The equality case is handled by the results of Tod [28]; the plane waves case allowed by Tod is excluded by [4, Theorem 3.4]. We note that in the $\hat{m} = |\hat{Q}|$ case it is not known whether one can conclude that the metric must be (locally or globally) a standard Majumdar–Papapetrou metric.

To proceed further, we need to analyze the behavior of h and ϕ near $\partial\Sigma$. We shall give here an overview of the results needed, and we refer the reader to [8] for detailed proofs of the results discussed in this paragraph. Recall, thus, that by the Vishveshwara–Carter Lemma [5, 29] $\partial\Sigma$ must be a subset of (the closure of) a Killing horizon $\overline{\mathcal{N}}$. By that same lemma one knows that in a static space–time the Killing horizon is a smooth submanifold. Standard results [18] show that ϕ is constant on any connected component of $\overline{\mathcal{N}}$, hence of $\partial\Sigma$. A connected component S of $\partial\Sigma$ will be called degenerate, respectively non–degenerate, if S intersects a degenerate, respectively non–degenerate Killing horizon. By deforming Σ slightly in space–time if necessary we can ensure that $\partial\Sigma$ is a smooth submanifold both of $\bar{\Sigma}$ and of M near degenerate horizons. Every degenerate component corresponds to a complete end of (Σ, h) [8, Prop. 3.2]. As far as non–degenerate horizons are concerned, $\partial\Sigma$ will not be a smooth submanifold of M in general when there are points on $\partial\Sigma$ at which the Killing vector field X vanishes. However we can equip $\bar{\Sigma}$ with a differentiable structure so that $\partial\Sigma$ is a smooth submanifold of $\bar{\Sigma}$ [8, Prop. 3.3]. Moreover $\partial\Sigma$ with this differentiable structure is a totally geodesic boundary of (Σ, h) across which h can be extended smoothly when doubling Σ. Now ϕ is a smooth function on space–time, and the proof of [8, Prop. 3.3] shows that ϕ is a smooth function of (x^2, y^a) (here x^2 denotes the square of x, and not an index 2 on x) in an appropriate coordinate system near a non–degenerate connected component S of $\partial\Sigma$, with S given by $x = 0$ in this coordinate system. This implies in particular that

$$|d\phi|_h(x = 0) = 0 \ , \tag{3.11}$$

and that ϕ extends smoothly across S when a doubling of Σ across S is performed.

We have the following, which is based on an observation of Ruback [25]:

Proposition 3.4 *Under the hypotheses of Theorem 1.1 we have*

$$0 \leq V + |\phi| \leq 1 \tag{3.12}$$

on $\bar{\Sigma}$, with the inequalities being strict on Σ except if the metric is, locally, a Majumdar–Papapetrou metric. Further the right inequality is strict on non–degenerate horizons.

PROOF: Set

$$F_\pm = V^2 - (1 \pm \phi)^2 \ ;$$

as noted by Ruback [25] the functions $F_\pm$ satisfy the equation

$$\Delta_\gamma F_\pm = 0 \ , \tag{3.13}$$

where Δ_γ is the Laplace operator of the metric $V^{-2}h_{ij}$. In the asymptotically flat region of Σ_{ext} the $F_\pm$'s approach zero, while at every component of $\partial\Sigma$ we have $F_\pm \le 0$.

Suppose, first, that $F_- = 0$ on all components of $\partial\Sigma$; the maximum principle implies then

$$F_- \equiv 0$$

on $\bar{\Sigma}$. Equation (3.8) and the transformation rule of the Ricci tensor under conformal transformations show that the metric $(1-\phi)^{-2}h_{ij}$ is Ricci flat. In dimension three this implies flatness, and the proof of Lemma 5.1 of [8] shows that near Σ the space–time metric can locally be written in the Majumdar–Papapetrou form (2.7). A similar analysis applies if F_+ vanishes throughout $\partial\Sigma$.

It remains to consider the case in which both F_+ and F_- are negative somewhere on $\partial\Sigma$. From the maximum principle one obtains

$$F_\pm < 0 \tag{3.14}$$

on Σ, so that

$$V^2 < (1-\phi)^2 \ , \qquad V^2 < (1+\phi)^2 \tag{3.15}$$

on Σ. V has no zeros on Σ by hypothesis, which together with (3.15) shows that both $1-\phi$ and $1+\phi$ have no zeros on Σ. As both $1-\phi$ and $1+\phi$ go to 1 at the infinity of Σ_{ext} it follows that

$$-1 < \phi < 1 \tag{3.16}$$

on Σ. Equations (3.15)–(3.16) imply $0 < V < \min(1+\phi, 1-\phi) = 1 - |\phi|$ on Σ, as desired.

It remains to consider what happens on non–degenerate components of $\partial\Sigma$. Let, thus, S be a connected non–degenerate component of $\partial\Sigma$, so that $d\phi$ vanishes on S by Equation (3.11). It is well known, and in any case easily checked from the formulae in [8, Section 3], that

$$|dV|_h(x=0) = \kappa \ , \tag{3.17}$$

where κ is the surface gravity of S; the condition that S is non–degenerate is precisely $\kappa \neq 0$. Suppose that $\phi = 1$ on S, then F_- vanishes on S and Equation

(3.17) shows that $F_- = \kappa^2 x^2 + O(x^4)$ will be positive in a neighborhood of S (recall that $\phi - \phi|_S = O(x^2)$), which contradicts (3.14). Similarly $\phi = -1$ on S would lead to F_+ being positive in a neighborhood of S, again a contradiction. ■

We note the following corollary[6] of Proposition 3.4:

Corollary 3.5 *Under the hypotheses of Theorem 1.1 we have*

$$m \geq |Q| \ . \tag{3.18}$$

where $m > 0$ is the ADM mass of $\Sigma_{\rm ext}$ and Q the total charge of $\Sigma_{\rm ext}$. Further, if the inequality is attained the metric is, locally, a Majumdar–Papapetrou metric.

PROOF: A theorem of Beig [3] (*cf.* also [2, 6]) shows that the Komar mass of a static asymptotically vacuum end $(\Sigma_{\rm ext}, g|_{\Sigma_{\rm ext}})$ coincides with its ADM mass, so that we have

$$V = 1 - \frac{m}{r} + O(r^{-2}) \ . \tag{3.19}$$

The inequality (3.18) follows immediately from Proposition 3.4 and the asymptotic expansion (3.4). If $m = Q$ we have $F_+ = O(r^{-2})$, and $F_+ \equiv 0$ follows from (3.13) and the asymptotic strong maximum principle of [27, Appendix]. The conclusion that the metric is locally a Majumdar–Papapetrou metric follows then as in the proof of Proposition 3.4. The case $m = -Q$ follows similarly by considering F_-. The inequality $m > 0$ follows either from the asymptotic strong maximum principle of [27, Appendix] or from [16]. ■

It follows from Proposition 3.4 that ϕ satisfies the inequality $-1 < \phi < 1$ on Σ, and that the values $\phi = 1$ or $\phi = -1$ can only be attained at degenerate components of $\partial\Sigma$. When only one component of the event horizon is degenerate we can without loss of generality assume, changing ϕ to $-\phi$ if necessary, that we have

$$-1 < \phi \leq 1 \text{ on } \bar{\Sigma} \ . \tag{3.20}$$

It is tempting to conjecture that one can *always* assume, changing ϕ to $-\phi$ if necessary, that

$$0 \leq \phi \leq 1 \text{ on } \bar{\Sigma} \ .$$

This is due to the fact that a change of the sign of ϕ will necessarily lead to both positive and negative charges of event horizons, *cf.* Lemma 3.7 below — such a configuration is unlikely to be static. Whatever the situation, if Equation (3.20) holds we can prove the following:

[6]The inequality (3.18) has been established under rather more general circumstances in [16, Remark, p. 107], using a technique suggested by Gibbons and Hull [11].

Theorem 3.6 *Let (M, g, F) be a static solution of the Einstein–Maxwell equations with defining Killing vector X. Suppose that M contains a connected and simply connected space–like hypersurface Σ the closure $\bar{\Sigma}$ of which is the union of an asymptotically flat end and of a compact interior, such that:*

1. *We have $g_{\mu\nu}X^\mu X^\nu < 0$ on Σ.*
2. *The topological boundary $\partial\Sigma \equiv \overline{\Sigma} \setminus \Sigma$ of Σ is a nonempty topological manifold, with $g_{\mu\nu}X^\mu X^\nu = 0$ on $\partial\Sigma$.*

If Equation (3.20) holds, then the conclusions of Theorem 1.1 hold.

Proof: The case $m = |Q| = 0$ cannot occur by [16]. If $m = |Q| \neq 0$ the metric is, locally, of Majumdar–Papapetrou form by Corollary 3.5. In that case we can apply [**?**, Theorem 7.2] (*cf.* also [9]) to the Killing development $(\hat{M}, \hat{g})$ of Σ as defined in [4] to conclude that $(\hat{M}, \hat{g})$ is a standard Majumdar–Papapetrou space–time, and the result follows; *cf.* the argument around Equation (3.35) below for a more detailed exposition of the construction of the embedding in the Reissner–Nordström context.

It remains to analyze the case $m > |Q|$. In order to do that, consider the manifold Σ equipped with the metric h defined by Equation (3.1). From what has been said (Σ, h) is a complete Riemannian manifold with compact (perhaps empty) boundary and with at least one asymptotically flat end $\Sigma_{\rm ext}$. Let us denote by $\partial_{nd}\Sigma$ the collection of all those components of the boundary of Σ which correspond to non–degenerate components of the event horizon of the black hole. Following [25], if $\partial_{nd}\Sigma \neq \emptyset$ we set

$$\Sigma_+ = \Sigma, \qquad h_+ = \left(\tfrac{1+V+\phi}{2}\right)^2 h \ ,$$

$$\Sigma_- = \Sigma, \qquad h_- = \left(\tfrac{1-V+\phi}{2}\right)^2 h \ ,$$

$$\hat{\Sigma} = \Sigma_+ \cup \Sigma_- \cup \partial_{nd}\Sigma \ , \qquad \hat{h}\Big|_{\Sigma_+} = h_+ \ , \quad \hat{h}\Big|_{\Sigma_-} = h_- \ , \tag{3.21}$$

$$\hat{E}_\pm = \frac{(1+\phi)d\phi - VdV}{V(1+\phi\pm V)} \ , \qquad \hat{E}\Big|_{\Sigma_+} = \hat{E}_+ \ , \quad \hat{E}\Big|_{\Sigma_-} = \hat{E}_- \ . \tag{3.22}$$

The topological and differentiable structure of $\hat{\Sigma}$ are defined through the gluing of $\overline{\Sigma}_+ \equiv \Sigma_+ \cup \partial_{nd}\Sigma$ with $\overline{\Sigma}_- \equiv \Sigma_- \cup \partial_{nd}\Sigma$ by identifying $\partial_{nd}\Sigma$, considered as a subset of $\overline{\Sigma}_+$, with a second copy of $\partial_{nd}\Sigma$, considered as a subset of $\overline{\Sigma}_-$, using the identity map. From our remarks at the beginning of this section it follows that the metric $\hat{h}$ defined on $\Sigma_+ \cup \Sigma_-$ in (3.21) can be extended by continuity to a smooth metric on $\hat{\Sigma}$; similarly $\hat{E}$ can be extended by continuity to a smooth vector field on $\hat{\Sigma}$.

If $\partial_{nd}\Sigma = \emptyset$ we set

$$\hat{\Sigma} = \Sigma \ , \qquad \hat{h} = h_+ \ , \qquad \hat{E} = E_+ \ .$$

We have the following:

- The conformal factor $1-V+\phi \geq 1-V-|\phi|$ is strictly positive on $\Sigma \cup \partial_{nd}\Sigma$ by Proposition 3.4, and so is $1+V+\phi = 2V+1-V+\phi \geq 1-V+\phi$. Near every connected degenerate component S of $\partial\Sigma$ the electric potential ϕ will tend to a value different from -1 by the hypothesis (3.20), while V will tend to zero, hence the asymptotic end of (Σ, h) corresponding to S remains complete in the metric $(\hat{\Sigma}, \hat{h})$.

- The conformal factor $(1+V+\phi)/2$ tends to 1 in the asymptotically flat end $\Sigma_{\rm ext}$, so that $\Sigma_{\rm ext}$ is an asymptotically flat end for the metric h_+, with ADM mass equal to
$$\hat{m} = \frac{1}{2}(m+Q) \ .$$
The electric field $\hat{E}$ approaches zero as $1/r^2$ in $\Sigma_{\rm ext}$ and has charge equal to
$$\hat{Q} = \frac{1}{2}(m+Q) = \hat{m} \ .$$

- The conformal factor $1-V+\phi$ tends to 0 in the asymptotically flat $\Sigma_{\rm ext}$ as $(m+Q)/r$, with $m+Q \neq 0$, thus as r tends to infinity in $\Sigma_{\rm ext}$ the metric h_- approaches, to leading significant orders, the metric
$$\frac{(m+Q)^2}{4}\Big(\frac{1}{r^2}dr^2 + d\Omega^2\Big) \ ,$$
where $d\Omega^2$ is the standard round metric on a two sphere. It easily follows that $(\Sigma_{\rm ext}, h_-)$ is a *complete* end of $(\hat{\Sigma}, \hat{h})$ (*cf.* the calculation in the proof of Proposition 3.2 in [8]).

As emphasized by Ruback [25] we have
$$\hat{R} = 2\hat{h}(\hat{E},\hat{E}) \ , \tag{3.23}$$
$$\hat{\nabla}_i \hat{E}^i = 0 \ , \tag{3.24}$$
where $\hat{\nabla}$ is the covariant derivative of the metric $\hat{h}$. Thus the hypotheses of Theorem 3.3 are satisfied. Since the mass of $\hat{h}$ and the charge of $\hat{E}$ coincide, Theorem 3.3 shows that $\hat{h}$ is, locally, the space part of the Majumdar–Papapetrou[7] metric. This shows in particular that $\hat{h}$, and hence also h, are conformally flat, so that the Cotton tensor B_{ijk} of h satisfies
$$B_{ijk} \equiv 0 \ . \tag{3.25}$$

[7]It might be worthwhile to point out that it is not known at this stage that $\hat{h}$ is the space part of a standard Majumdar–Papapetrou metric, but this information is not needed in the argument.

Equation (3.10) implies $d\hat{E} = 0$ and from Equation (3.22) we have

$$0 = d\hat{E} = 2\frac{d\phi \wedge dV}{V^2} \ . \tag{3.26}$$

It follows that $d\phi$ is parallel to dV wherever dV does not vanish. Standard results about solutions of elliptic equations show that

$$dV = -\frac{mdr}{r^2} + O(r^{-3}) \ , \tag{3.27}$$

so that dV does not vanish for $r \geq R$, for an R large enough. Increasing R if necessary it follows from Equation (3.19) and from the maximum principle that for $r \geq R$ the level sets of V will be embedded spheres. One also finds that there exists $0 \leq V_- < 1$ such that for $c \in [V_-, 1)$ the level sets $\{V = c\}$ are smooth embedded spheres. Let

$$\begin{aligned} \hat{I} &= \{c \mid c \text{ is a non–critical value of } V\} \ , \\ \hat{\mathcal{U}} &= \{p \mid V(p) \in \hat{I}\} = \cup_{c\in\hat{I}} V^{-1}(c) \ , \end{aligned}$$

and define $\mathcal{U}$ to be that connected component of $\hat{\mathcal{U}}$ that contains $\mathbb{R}^3 \setminus B(0, R)$. (Recall that c is non–critical if dV is nowhere vanishing on the level set $V = c$.) Similarly define $I \subset (0, 1)$ to be that connected component of $\hat{I}\setminus\{0\}$ that contains $(V_-, 1)$; clearly

$$\mathcal{U} = \{p \mid V(p) \in I\} = \cup_{c\in I} V^{-1}(c) \ .$$

Compactness of the level sets of V implies that $\mathcal{U}$ is diffeomorphic to $I \times S^2$, and that on $\mathcal{U}$ the function V can be used as a coordinate. Further we can introduce a finite number of coordinate patches with coordinates x^A, $A = 1, 2$, on S^2 so that on $\mathcal{U}$ the metric takes the form

$$h = W^{-2}dV^2 + \gamma_{AB}dx^A dx^B \ . \tag{3.28}$$

Equation (3.26) shows that

$$\phi = \phi(V)$$

on $\mathcal{U}$. This allows one to write Equations (3.5) and (3.7) in the coordinate system (3.28) as

$$\frac{\partial\phi}{\partial V} = \frac{V}{W\sqrt{\det\gamma_{AB}}}\frac{\partial}{\partial V}\Big(W\sqrt{\det\gamma_{AB}}\frac{\partial\phi}{\partial V}\Big) \ , \tag{3.29}$$

$$\Big(\frac{\partial\phi}{\partial V}\Big)^2 = \frac{V}{W\sqrt{\det\gamma_{AB}}}\frac{\partial}{\partial V}\Big(W\sqrt{\det\gamma_{AB}}\Big) \ . \tag{3.30}$$

It follows that

$$V\frac{\partial^2\phi}{\partial V^2} = \frac{\partial\phi}{\partial V} - \Big(\frac{\partial\phi}{\partial V}\Big)^3 \ .$$

Integrating this equation one finds

$$\frac{\partial \phi}{\partial V} = \frac{AV}{\sqrt{1 + A^2 V^2}} \ , \tag{3.31}$$

where A is an integration constant. From Equation (3.27), from $m > 0$ and from $d\phi = -Qdr/r^2 + O(r^{-3})$ one obtains

$$\lim_{V\to 1} \frac{\partial \phi}{\partial V} = -\frac{Q}{m} \ ,$$

which determines A.

Suppose first that $Q = 0$, then

$$d\phi \equiv 0$$

on $\mathcal{U}$ by Equation (3.31). Equation (3.5) and the unique continuation theorem of Aronszajn [1] show that ϕ is constant throughout Σ, so that the initial data set is vacuum. In this case the space–time metric is the Schwarzschild metric in a neighborhood of Σ by [8, Theorem 1.1].

It remains to consider the case $Q \neq 0$. Integrating Equation (3.31) and using $\lim_{V\to 1} \phi = 0$ we obtain

$$\phi = \frac{m - \sqrt{m^2 + Q^2(V^2 - 1)}}{Q} \ . \tag{3.32}$$

According to Heusler [18, Equation (9.58)] (*cf.* also [23]) this implies

$$\frac{V^4}{8W^4} B_{ijk} B^{ijk} = \Big(\frac{m^2 - Q^2}{m^2 + Q^2(V^2 - 1)} \Big)^2 \Big(|\lambda|^2_\gamma + \frac{|\mathcal{D}W|^2_\gamma}{2} \Big) \ .$$

Here $|\cdot|_\gamma$ denotes the norm with respect to the metric $\gamma = \gamma_{AB} dx^A dx^B$, $\mathcal{D}W$ is the gradient of the restriction of the function W (defined in (3.28)) to the level sets of V, and $\lambda \equiv \lambda_{AB} dx^A dx^B$ is the trace free part of the extrinsic curvature tensor of the level sets of V — in the coordinate system of (3.28)

$$\lambda_{AB} = W \Big(\frac{\partial \gamma_{AB}}{\partial V} - \frac{1}{2} \gamma^{CD} \frac{\partial \gamma_{CD}}{\partial V} \gamma_{AB} \Big) \ . \tag{3.33}$$

Equation (3.25) implies that $\frac{\partial \gamma_{AB}}{\partial V}$ is pure trace, and that $W = W(V)$. This latter property and Equation (3.30) show that $\det \gamma_{AB}$ is a product of a function of V with a function of the remaining coordinates. From the asymptotic behavior of the metric it then follows that

$$h = W(V)^{-2} dV^2 + H(V) d\Omega^2 \ . \tag{3.34}$$

for some function $H(V)$, where $d\Omega^2$ is the standard round metric on S^2. A straightforward integration of Equations (3.30) and (3.9) using (3.32) shows that

the metric on $\mathcal{U}$ is the space part of the Reissner–Nordström metric. In other words, h is on $\mathcal{U}$ the pull back by a suitable diffeomorphism ψ of the space part $h^{\rm RN}$ of the Reissner–Nordström metric.

To finish the proof[8], we claim that I is open in $(0,1)$, which can be seen as follows: Let $p \in \mathcal{U}$, we thus have $dV(q) \neq 0$ for all q such that $V(p) = V(q)$. By Equation (3.34) $|dV|_h = W$ is constant on the level set $V^{-1}(V(p))$ of V through p so that

$$\inf_{V^{-1}(V(p))} |dV|_h > 0 \ ,$$

which easily implies that all nearby level sets are non–critical.

To see that I is closed in $(0,1)$, recall that, using obvious notation, we have $h = \psi^* h^{\rm RN}$ and $V = V^{\rm RN} \circ \psi$ on $\mathcal{U}$. Let $s_i \in I$ be any sequence converging to $s \in (0,1)$, thus $s_i = V(p_i)$ for some $p_i \in \mathcal{U}$. By the interior compactness of Σ, passing to a subsequence if necessary, there exists $p \in \Sigma$ such that $p_i \to p$, with $V(p) = s > 0$. Set

$$C = \inf |dV^{\rm RN}|_{h^{\rm RN}} \ ,$$

where the infimum is taken over those points q in $M^{\rm RN}$ for which $V^{\rm RN}(q) > V(p)/2$. We have $V(p_i) = s_i > V(p)/2$ for i large enough, so that $V^{\rm RN}(\psi(p_i)) = V(p_i) > V(p)/2$. It follows that

$$|dV|_h(p) = \lim_{i\to\infty} |dV|_h(p_i) = \lim_{i\to\infty} |dV^{\rm RN}|_{h^{\rm RN}}(\psi(p_i)) > C \ ,$$

so that $dV(p) \neq 0$. Now $|dV|_h$ is constant on those level sets of V which are in I, and by continuity it is also constant on those level sets of V which are in $\bar{I}$, the closure of I in $(0,1)$. Hence $|dV|_h$ is non–vanishing on the level set $\{V = s\}$, thus $s \in I$.

We have thus shown that I is open and closed, and connectedness of Σ implies $\mathcal{U} = \Sigma$. Thus the manifold $\mathbb{R} \times \Sigma$ with the metric $-V^2 dt^2 + h$ is isometrically diffeomorphic to the Reissner–Nordström space–time.

Consider any neighborhood $\mathcal{V}$ of Σ diffeomorphic to an open interval times Σ; the set $\mathcal{V}$ is simply connected by simple–connectedness of Σ. Let α be the one–form

$$\alpha = \frac{X_\mu dx^\mu}{X_\nu X^\nu} \ ;$$

Equation (2.1) shows that α is closed, and simple–connectedness of $\mathcal{V}$ implies existence of a function $t \in C^\infty(\mathcal{V})$ such that $\alpha = dt$. As in the proof of Lemma 5.1 of [8] there exists a function $f : \Sigma \to \mathbb{R}$ such that

$$t = s + f \ , \qquad (3.35)$$

[8]We note that it is usual at this stage to invoke analyticity to conclude the proof. Because analytic extensions of manifolds are not unique this is not sufficient without further justification. The argument we present here avoids this problem.

Here s denotes the coordinate along the (perhaps only locally defined) orbits of the Killing vector field on $\mathcal{V}$. Passing to a subset of $\mathcal{V}$ if necessary we may assume that every orbit of X in $\mathcal{V}$ intersects Σ precisely once. We can then extend f to a function on $\mathcal{V}$ by requiring that $X(f) = 0$. As the metric $-V^2dt^2 + h$ has already been shown to be the Reissner–Nordström metric, Equation (3.35) provides now the required embedding of $\mathcal{V}$ into an open subset of the Reissner–Nordström space–time. ■

In order to show that Theorem 1.1 is a special case of Theorem 3.6 we need the following result:

Lemma 3.7 *Let S_a, $a = 1, 2$ be connected components of $\partial\Sigma$ such that the horizon potentials $\phi_a = \phi|_{S_a}$ satisfy*

$$\phi_1 = \inf_{\bar{\Sigma}} \phi < 0 \ , \qquad \phi_2 = \sup_{\bar{\Sigma}} \phi > 0 \ . \tag{3.36}$$

Then the charges Q_a of the S_a's are non–vanishing and have opposite signs.

The result it obtained by standard integration by parts arguments. However, some care must be taken in our context because the degenerate components of the boundary $\partial\Sigma$ lie at infinite h–distance, and because V tends to zero there.

Proof: Recall that the charges of the S_a's can be defined by the equations

$$Q_a = -\lim_{i\to\infty} \int_{S_{a,i}} V^{-1}\nabla^i\phi \, dS_i \ , \tag{3.37}$$

where the $S_{a,i}$ are any family of connected smooth hypersurfaces converging in an appropriate sense to the S_a's as i tends to infinity. For definiteness:

- If S_a is degenerate we take the $S_{a,i}$'s to be the sets $x = 1/i$, where x is the coordinate of the proof of Proposition 3.2 of [8], and we assume that x has been rescaled so that its range covers the interval $[0, 1]$; we set
$$\Omega_a = \{x < 1\} \ .$$
- If S_a is non–degenerate we take the $S_{a,i}$'s to be the sets $w = 1/i$, where w is the coordinate of the proof of Proposition 3.3 of [8], and we assume that w has been rescaled so that its range covers the interval $[0, 1]$; we set
$$\Omega_a = \{w < 1\} \ .$$

The integrals at the right–hand–side of Equation (3.37) are i independent by equation (3.5) and the divergence theorem,

$$\int_{S_{a,i}} V^{-1}\nabla^i\phi \, dS_i - \int_{S_{a,j}} V^{-1}\nabla^i\phi \, dS_i = \int_{\mathcal{V}_{i,j}} \nabla_i(V^{-1}\nabla^i\phi) \, d\mu_h = 0 \ .$$

Here $\mathcal{V}_{i,j}$ is the volume the boundary of which consists of $S_{a,i}$ and $S_{a,j}$. Hence the limit in (3.37) exists.

Let

$$\phi_- = \inf_{S_{1,1}} \phi \ , \qquad \phi_+ = \sup_{S_{2,1}} \phi \ .$$

By Equation (3.36) and the maximum principle we have $\phi_1 < \phi_-$ and $\phi_+ < \phi_2$. Let c be a non–critical value of ϕ satisfying $\phi_1 < c < \phi_-$, then the level set $\phi^{-1}\{c\} \cap \Omega_1$ is a smooth compact submanifold of Ω_1; recall that the set of non–critical values of ϕ is dense by Sard's theorem (*cf., e.g.,* [13]). Applying the divergence theorem on a set bounded by $\phi^{-1}\{d\} \cap \Omega_1$ (with a non–critical d satisfying $\phi_1 < d < \phi_-$) and by $S_{1,i}$ for an i large enough we obtain

$$\int_{\phi^{-1}\{d\}\cap\Omega_1} V^{-1}\nabla^i\phi \, dS_i = -Q_1 \ . \tag{3.38}$$

Let c and d be any non–critical values of ϕ satisfying $\phi_1 < c < d < \phi_-$, thus $\mathcal{W}_{cd} \equiv \{c \le \phi \le d\} \cap \Omega_1$ is a smooth compact submanifold of Ω_1 with boundary $(\phi^{-1}\{c\} \cup \phi^{-1}\{d\}) \cap \Omega_1$. By the maximum principle and the boundary point lemma [12, Lemma 3.4] we have

$$h(\nabla\phi, n) > 0$$

on $\phi^{-1}\{d\})\cap\Omega_1$, where n is that unit normal to $\phi^{-1}\{d\}\cap\Omega_1$ which points outwards from $\mathcal{W}_{cd}$, hence

$$Q_1 = -\int_{\phi^{-1}\{d\}\cap\Omega_1} V^{-1}\nabla^i\phi \, dS_i < 0 \ .$$

The inequality $Q_2 > 0$ follows by changing ϕ to $-\phi$ in the argument above. ∎

We can now pass to the

PROOF OF THEOREM 1.1: If $\partial\Sigma$ is connected the hypotheses of Theorem 3.6 are obviously satisfied, and the result follows. Suppose, thus, that $\partial\Sigma$ is not connected. Changing ϕ to $-\phi$ if necessary we will be able to satisfy (3.20) unless there exists a connected component S_1 of $\partial\Sigma$ such that $\phi_1 = -1$ and a connected component S_2 of $\partial\Sigma$ such that $\phi_2 = 1$. By Proposition 3.4 S_1 and S_2 have to be degenerate, and by Lemma 3.7 the charges of S_1 and S_2 have opposite signs. This is, however, not allowed by the hypotheses of Theorem 1.1, and the result follows by Theorem 3.6. ∎

We finally note that the Reissner–Nordström case of Corollary 1.2 is proved by a repetition of the arguments of the proof of Corollary 1.2 in [8]. The Majumdar–Papapetrou case is proved by a repetition of the arguments of the proof of Corollary 1.2 in [8] together with the arguments presented in the first paragraph of the proof of Theorem 3.6.

References

[1] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, Jour. de Math. **XXXVI** (1957), 235–249.

[2] A. Ashtekar and A. Magnon-Ashtekar, *On conserved quantities in general relativity*, Jour. Math. Phys. **20** (1979), 793–800.

[3] R. Beig, *Arnowitt–Deser–Misner energy and* g_{00}, Phys. Lett. **69A** (1978), 153–155.

[4] R. Beig and P.T. Chruściel, *Killing vectors in asymptotically flat space-times: I. Asymptotically translational Killing vectors and the rigid positive energy theorem*, Jour. Math. Phys. **37** (1996), 1939–1961, arXiv:gr-qc/9510015.

[5] B. Carter, *Killing horizons and orthogonally transitive groups in space-time*, Jour. Math. Phys. **10** (1969), 70–81.

[6] P.T. Chruściel, *A remark on the positive energy theorem*, Class. Quantum Grav. **33** (1986), L115–L121.

[7] ______, *"No Hair" Theorems – folklore, conjectures, results*, Differential Geometry and Mathematical Physics (J. Beem and K.L. Duggal, eds.), Cont. Math., vol. 170, AMS, Providence, 1994, pp. 23–49, arXiv:gr–qc/9402032.

[8] ______, *The classification of static vacuum space-times containing an asymptotically flat spacelike hypersurface with compact interior*, Class. Quantum Grav. **16** (1999), 661–687, *Corrigendum* in arXiv:gr-qc/9809088v2.

[9] P.T. Chruściel and N.S. Nadirashvili, *All electrovacuum Majumdar–Papapetrou space-times with non–singular black holes*, Class. Quantum Grav. **12** (1995), L17–L23, arXiv:gr-qc/9412044.

[10] P.T. Chruściel and R.M. Wald, *On the topology of stationary black holes*, Class. Quantum Grav. **11** (1994), no. 12, L147–152, arXiv:gr–qc/9410004. MR MR1307013 (95j:83080)

[11] G.W. Gibbons and C.M. Hull, *A Bogomolny bound for general relativity and solitons in* $N = 2$ *supergravity*, Phys. Lett. **109B** (1982), 190–194.

[12] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 1983.

[13] V. Guillemin and A. Pollack, *Differential topology*, Prentice–Hall, Englewood Cliffs, N.J, 1974.

[14] J.B. Hartle and S.W. Hawking, *Solutions of the Einstein–Maxwell equations with many black holes*, Commun. Math. Phys. **26** (1972), 87–101.

[15] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, Cambridge, 1973, Cambridge Monographs on Mathematical Physics, No. 1. MR MR0424186 (54 #12154)

[16] M. Herzlich, *The positive mass theorem for black holes revisited*, Jour. Geom. Phys. **26** (1998), 97–111.

[17] M. Heusler, *On the uniqueness of the Reissner–Nordström solution with electric and magnetic charge*, Class. Quantum Grav. **11** (1994), L49–L53.

[18] ______, *Black hole uniqueness theorems*, Cambridge University Press, Cambridge, 1996.

[19] ______, *On the uniqueness of the Papapetrou–Majumdar metric*, Class. Quantum Grav. **14** (1997), L129–L134, arXiv:gr-qc/9607001.

[20] W. Israel, *Event horizons in static electrovac space-times*, Commun. Math. Phys. **8** (1968), 245–260.

[21] S.D. Majumdar, *A class of exact solutions of Einstein's field equations*, Phys. Rev. **72** (1947), 390–398.

[22] A.K.M. Masood–ul–Alam, *Uniqueness proof of static charged black holes revisited*, Class. Quantum Grav. **9** (1992), L53–L55.

[23] H. Müller zum Hagen, D.C. Robinson, and H.J. Seifert, *Black holes in static electrovac space-times*, Gen. Rel. Grav. **5** (1974), 61–72.

[24] A. Papapetrou, *A static solution of the equations of the gravitational field for an arbitrary charge distribution*, Proc. Roy. Irish Acad. **A51** (1947), 191–204.

[25] P. Ruback, *A new uniqueness theorem for charged black holes*, Class. Quantum Grav. **5** (1988), L155–L159.

[26] W. Simon, *Radiative Einstein-Maxwell spacetimes and 'no–hair' theorems*, Class. Quantum Grav. **9** (1992), 241–256.

[27] D. Sudarsky and R.M. Wald, *Extrema of mass, stationarity and staticity, and solutions to the Einstein–Yang–Mills equations*, Phys. Rev. **D46** (1993), 1453–1474.

[28] K.P. Tod, *All metrics admitting super–covariantly constant spinors*, Phys. Lett. **121B** (1983), 241–244.

[29] C.V. Vishveshwara, *Generalization of the "Schwarzschild Surface" to arbitrary static and stationary metrics*, Jour. Math. Phys. **9** (1968), 1319–1322.